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Gauge transformation for dynamical systems of Ising spin glasses

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Abstract. Dynamical systems of gauge-symmetric Ising spin glasses are investigated by the method of gauge transformation. Several exact relations are derived among dynamical quantities such as the equilibrium autocorrelation function and the non-equilibrium remanent magnetization. The same result as in the static case is obtained in terms of the equivalence of the ferromagnetic and the spin-glass order if the temperature and the randomness satisfy a special condition (Nishimori line). An exact equivalence of non-equilibrium relaxations in the spin-glass phase is derived between the remanent magnetization evolved from the strong-field limit and the autocorrelation function from a supercooled state. We also have a plausible argument for the absence of a re-entrant transition using the present dynamical relations.

1. Introduction

In the last decade, numerous studies have been made on the theory of short-range spin glasses [1, 2]; in particular, the Edwards–Anderson-type Ising spin-glass models have attracted much attention. In the $\pm J$ model with symmetric bond distributions (the concentration of the ferromagnetic bonds is $\frac{1}{2}$), the existence of the spin-glass (SG) phase has been confirmed in three dimensions [3, 4]. In the asymmetric case, in which paramagnetic PM, ferromagnetic (FM) and SG phases appear, the phase diagram has been obtained in two and three dimensions [5–13].

The method of gauge transformation [14–16] is a powerful technique for deriving analytic results in the $\pm J$ or the Gaussian Ising spin glasses. It provides the internal energy and an upper bound on the specific heat exactly as non-singular functions of the temperature on a special line in the randomness–temperature phase diagram. This line is called the Nishimori line [14]. Further, it is proven that the boundary between the FM phase and non-FM one (SG in 3D) in the temperature lower than the multicritical point, at which the Nishimori line intersects, is vertical or re-entrant. A typical phase diagram in the randomness–temperature plane is shown in figure 1. Kitatani [15] introduced a model with a slightly different bond distribution, and showed the verticality which implies the absence of re-entrant transition; although he made a plausible but unproved assumption about the thermodynamic behaviour of the modified model, the result is consistent with previous theories [6–8, 10, 11, 13] and experiments on Ising-like spin glasses [17–19]. Recently, the gauge transformation has been applied to gauge glasses with various symmetries [16].

With all such important results, the method of gauge transformation has not yet been applied to dynamical systems. The dynamics is one of most important aspects in spin glasses because of the slow relaxation [1, 2]. In the mean-field theory, the non-ergodicity

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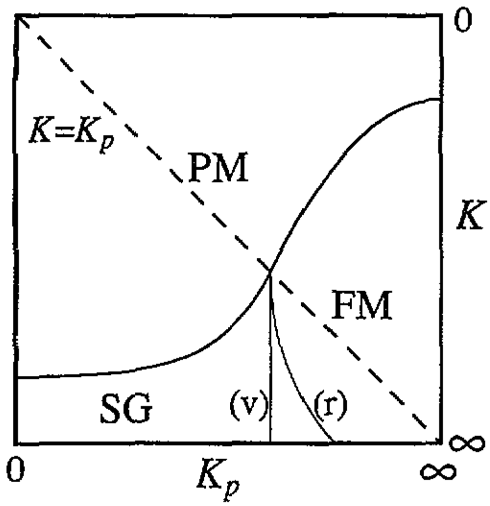

Figure 1. Typical phase diagram of Ising spin glasses in the K_p–K plane (K_p controls the randomness and $K = J/k_B T$). The broken line is the Nishimori line. Possible FM phase boundaries in low temperature region are indicated by dotted lines; (v) vertical and (r) re-entrant.

below the freezing temperature is significant to understand the nature of spin glasses [20]. In short-range systems, the study of dynamical properties [4] has been insufficient as compared with the static case. To investigate the generality of the mean-field field picture is one of most important problems in the theory of spin glasses. Thus, exact statements are highly desirable for the progress in this field.

In this paper, we study the gauge transformation for dynamical systems of Ising spin glasses. First, we define the stochastic dynamics of Ising spin glasses in section 2. Then, the gauge transformation is introduced for such dynamical systems in section 3. Using the exact dynamical relations obtained in section 3, we discuss the various properties of the system in section 4. Our purpose is (i) to prove the exact equivalence of non-equilibrium relaxations in the SG phase from two distinct initial states, the strong-field limit and a supercooled state, (ii) to examine the dynamics on the Nishimori line and confirm the consistency with the results obtained by the gauge transformation in the static case, and (iii) to show the absence of re-entrant transition from the FM phase to the SG. The last section is devoted to the summary.

2. Stochastic dynamics of Ising spin glasses

The Hamiltonian we consider is

$$\mathcal{H} = J\widetilde{\mathcal{H}}(S;\omega) = -J\sum_{\langle ij\rangle}\omega_{ij}S_iS_j \tag{2.1}$$

where S_i is an Ising spin taking $+1$ or -1, $S = (S_1, S_2, \ldots, S_N)$ represents a configuration of total N spins. The set $\omega = (\omega_{12}, \ldots)$ represents a configuration of total N_B bonds, and the summation is taken over all bonds; we make no restrictions on the type or the dimension of the lattice, whereas one may suppose usual nearest-neighbour interactions on the d-dimensional hypercubic lattice. The exchange interaction $J_{ij} = J\omega_{ij}$ is a random variable with a distribution such as the $\pm J$ or the Gaussian. For a particular bond configuration, the thermal distribution at the temperature $T = J/k_B K$ is defined by

$$\rho_e(S; K, \omega) = \frac{\exp\{-K\widetilde{\mathcal{H}}(S;\omega)\}}{Z(K,\omega)}. \tag{2.2}$$

The random average is denoted by

$$[\cdots]_{\rm c} = \sum_{\omega} P(\omega; K_p) \ldots . \tag{2.3}$$

The general form of the bond distribution in a gauge-symmetric model is expressed as

$$P(\omega; K_p) = \frac{D(\omega)}{Y(K_p)} \exp\left\{-K_p \widetilde{\mathcal{H}}(F; \omega)\right\} \tag{2.4}$$

where $F = (+, +, \ldots, +)$ represents the spin state with $S_i = +1$ for all i. In the $\pm J$ distribution, ω_{ij} takes $+1$ or -1, $p = (1 + {\rm e}^{-2K_p})^{-1}$ is the concentration of $+J$ bonds, $Y = (2\cosh K_p)^{N_{\rm B}}$ and $D = 1$. In the Gaussian distribution, ω_{ij} takes any real values, K_p is the centre of distribution J_0, $Y = {\rm e}^{N_{\rm B} {K_p}^2/2}$ and $D = {\rm e}^{-\sum \omega_{ij}^2/2}$ (the variance is set to be unity). We summarize these functions in table 1. In both distributions, $K_p = 0$ and ∞ correspond to the full random case and the non-random case, respectively. The Nishimori line is located on $K = K_p$ (see figure 1). In the following, for simplicity, we sometimes omit the dependence on spin set, bond set, inverse temperature K and/or randomness K_p from functions defined above, if they are trivial or unimportant.

Table 1. The summary of functions appeared in the bond distributions.

	$\pm J$	Gaussian
ω_{ij}	± 1	Any real value
K_p	$K_p = \frac{1}{2} \ln \dfrac{p}{1-p}$	$K_p = J_0$
$Y(K_p)$	$(2\cosh K_p)^{N_{\rm B}}$	$\exp\left(\frac{1}{2} N_{\rm B} {K_p}^2\right)$
$D(\omega)$	1	$\exp\left(-\frac{1}{2}\sum_{\langle ij \rangle} \omega_{ij}^2\right)$

Since the Ising system has no intrinsic dynamics, we consider a Markov process for a fixed bond configuration, in which the density of state evolves with the master equation [22]

$$\frac{\rm d}{{\rm d}t} \rho_t(S) = \sum_{S'} W(S|S') \rho_t(S') . \tag{2.5}$$

The solution of the master equation is given by

$$\rho_t(S) = \sum_{S'} \langle S|{\rm e}^{tW}|S'\rangle\, \rho_0(S') . \tag{2.6}$$

The matrix element $\langle S|{\rm e}^{tW}|S'\rangle$ plays a role of the conditional probability between two different times, and is defined by

$$\langle S|{\rm e}^{tW}|S'\rangle \equiv \sum_{n=0}^{\infty} \frac{t^n}{n!} \langle S|W^n|S'\rangle \tag{2.7}$$

$$\langle S|W^n|S'\rangle \equiv \sum_{S^{(2)}} \cdots \sum_{S^{(n)}} W(S|S^{(n)}) \prod_{k=2}^{n} W(S^{(k)}|S^{(k-1)}) \tag{2.8}$$

where $S^{(1)} = S'$. The matrix W is composed of non-negative off-diagonal elements, and satisfies two conditions,

$$W(S|S') \rho_{\rm e}(S'; K, \omega) = W(S'|S) \rho_{\rm e}(S; K, \omega) \tag{2.9}$$

$$\sum_{S'} W(S'|S) = 0 . \tag{2.10}$$

The former is called the detailed balance, and guarantees the stability of the equilibrium distribution ρ_e. The latter is necessary for the conservation of the probability. It has been proven that all eigenvalues of the matrix $\boldsymbol{W}$ with (2.9) and (2.10) are real and negative except the zero eigenvalue corresponding to the equilibrium distribution [22]. Thus, all solutions with any initial conditions tend to the equilibrium distribution as $t \to \infty$. In general, the matrix element $W(\boldsymbol{S}|\boldsymbol{S}')$ depends on the bond configuration $\boldsymbol{\omega}$. The conditions (2.9) and (2.10) are automatically satisfied by the expression with a symmetric matrix $w(\boldsymbol{S}|\boldsymbol{S}')$,

$$W(\boldsymbol{S}|\boldsymbol{S}') = \frac{w(\boldsymbol{S}|\boldsymbol{S}')}{\rho_e(\boldsymbol{S}')} - \delta(\boldsymbol{S}, \boldsymbol{S}') \sum_{\boldsymbol{S}''} \frac{w(\boldsymbol{S}''|\boldsymbol{S})}{\rho_e(\boldsymbol{S})}. \tag{2.11}$$

The matrix $w(\boldsymbol{S}|\boldsymbol{S}')$ depends on the detail of the dynamics such as

$$w(\boldsymbol{S}|\boldsymbol{S}') = w_0\, \delta_1[\boldsymbol{S}, \boldsymbol{S}']\, \rho_e(\boldsymbol{S})^{\theta(\Delta[\boldsymbol{S},\boldsymbol{S}'])} \rho_e(\boldsymbol{S}')^{\theta(\Delta[\boldsymbol{S}',\boldsymbol{S}])} \tag{2.12}$$

for the Metropolis dynamics [23], and

$$w(\boldsymbol{S}|\boldsymbol{S}') = w_0\, \delta_1[\boldsymbol{S}, \boldsymbol{S}'] \frac{\sqrt{\rho_e(\boldsymbol{S})\, \rho_e(\boldsymbol{S}')}}{\cosh\left(\frac{K}{2}\Delta[\boldsymbol{S}, \boldsymbol{S}']\right)} \tag{2.13}$$

for the Glauber dynamics [24], where $\theta(x)$ is the step function,

$$\delta_1[\boldsymbol{S}, \boldsymbol{S}'] \equiv \delta\left(1, \sum_i \frac{1 - S_i S_i'}{2}\right) \tag{2.14}$$

is the one-spin-flip operator and

$$\Delta[\boldsymbol{S}, \boldsymbol{S}'] \equiv \widetilde{\mathcal{H}}(\boldsymbol{S}) - \widetilde{\mathcal{H}}(\boldsymbol{S}') \tag{2.15}$$

is the energy difference of two states.

We consider the following dynamical functions. The equilibrium autocorrelation function $\left[\langle S_i(0) S_i(t)\rangle_K^{\mathrm{eq}}\right]_c$ is defined from

$$\langle S_i(0) S_i(t)\rangle_K^{\mathrm{eq}} \equiv \sum_{\boldsymbol{S},\boldsymbol{S}'} \langle \boldsymbol{S}|e^{t\boldsymbol{W}}|\boldsymbol{S}'\rangle\, S_i S_i'\, \rho_e(\boldsymbol{S}'; K, \boldsymbol{\omega}). \tag{2.16}$$

Note that $\langle\cdots\rangle_K^{\mathrm{eq}}$ in (2.16) represents the dynamical average in equilibrium not the static one $\langle\cdots\rangle_K$. In the thermodynamic limit, when $t \to \infty$, $\left[\langle S_i(0) S_i(t)\rangle_K^{\mathrm{eq}}\right]_c$ converges to the Edwards–Anderson order parameter q_{EA}, which is non-vanishing in FM or SG phases [1, 2]. We also define a non-equilibrium autocorrelation function $\left[\langle S_i(0) S_i(t)\rangle_K^{K'}\right]_c$ by

$$\langle S_i(0) S_i(t)\rangle_K^{K'} \equiv \sum_{\boldsymbol{S},\boldsymbol{S}'} \langle \boldsymbol{S}|e^{t\boldsymbol{W}}|\boldsymbol{S}'\rangle\, S_i S_i'\, \rho_e(\boldsymbol{S}'; K', \boldsymbol{\omega}). \tag{2.17}$$

It is different from the equilibrium one, equation (2.16); note that the temperature at $t = 0$ is $J/k_B K'$, while the system evolves with $J/k_B K$ in $t > 0$. Another quantity is the relaxation function of remanent magnetization $\left[\langle S_i(t)\rangle_K^F\right]_c$ defined by

$$\langle S_i(t)\rangle_K^F \equiv \sum_{\boldsymbol{S}} \langle \boldsymbol{S}|e^{t\boldsymbol{W}}|\boldsymbol{F}\rangle\, S_i \tag{2.18}$$

which is the local (site) magnetization at time t evolved from the complete FM state, $\boldsymbol{F} = (+, +, \cdots, +)$, at $t = 0$. When $t \to \infty$, the thermodynamic limit of $\left[\langle S_i(t)\rangle_K^F\right]_c$ approaches the spontaneous magnetization. Similar function for the exchange energy is defined in the same relaxation process as $\left[\langle \mathcal{H}(t)\rangle_K^F\right]_c$.

3. Gauge transformation for dynamical systems

In this section, we extend the technique of gauge transformation to dynamical systems. Let us first introduce the gauge transformation in static systems. The transformations for functions of S and ω are defined by

$$U_\sigma : S_i \longrightarrow S_i\sigma_i \tag{3.1}$$

$$V_\sigma : \omega_{ij} \longrightarrow \omega_{ij}\sigma_i\sigma_j \tag{3.2}$$

where $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)$ is an arbitrary state of N Ising spins. Both transformations form groups homomorphic to Z_2^N and $Z_2^{N_B}$, respectively; Z_2 is a cyclic group. The Hamiltonian $\widetilde{\mathcal{H}}(S;\omega)$ is invariant under the transformation $U_\sigma V_\sigma$,

$$U_\sigma V_\sigma \widetilde{\mathcal{H}}(S;\omega) = \widetilde{\mathcal{H}}(S;\omega)\,. \tag{3.3}$$

The bond distribution (2.4) is transformed as

$$V_\sigma P(\omega; K_p) = \frac{D(\omega)Z(K_p,\omega)}{Y(K_p)}\,\rho_e(\sigma; K_p, \omega) \tag{3.4}$$

for the $\pm J$ and the Gaussian distributions, since $D(\omega)$ is an even function of all ω_{ij} and satisfies [16]

$$V_\sigma D(\omega) = D(\omega). \tag{3.5}$$

Note that the spin set in ρ_e and the temperature are different from the usual ones. Another important property is the invariance of summation (or integral) for S and ω;

$$\sum_S \cdots = \sum_S U_\sigma \cdots \tag{3.6}$$

$$\sum_\omega \cdots = \sum_\omega V_\sigma \cdots\,. \tag{3.7}$$

Using equations (3.3), (3.4) and (3.6), one can show [16] that the partition function is gauge-invariant with respect to V_σ:

$$V_\sigma Z(K,\omega) = Z(K,\omega)\,. \tag{3.8}$$

Note that we use the terminology 'gauge invariant' only for the functions of the set ω invariant under V_σ.

As seen in the previous section, two different spin states, the initial state S' and the final state S, are necessary to describe the dynamical behaviour of the stochastic system. Thus, we introduce the transformation for S',

$$U'_\sigma : S'_i \longrightarrow S'_i\sigma_i\,. \tag{3.9}$$

To examine the gauge transformation of the functions defined above, we need to show the invariance

$$U_\sigma U'_\sigma V_\sigma\ \langle S|e^{tW}|S'\rangle = \langle S|e^{tW}|S'\rangle\,. \tag{3.10}$$

From equations (3.3)–(3.8), it is easy to see that $\rho(S; K, \omega)$, $\delta_1[S, S']$ and $\Delta[S, S']$ are invariant under $U_\sigma U'_\sigma V_\sigma$. This yields the invariance

$$U_\sigma U'_\sigma V_\sigma w(S|S') = w(S|S') \tag{3.11}$$

for the Metropolis (2.12) and the Glauber dynamics (2.13). Equations (2.11) and (3.11) yield

$$U_\sigma U'_\sigma V_\sigma W(S|S') = W(S|S') \tag{3.12}$$

where the invariance of summation for S'' like equation (3.6) is used in the derivation for the case $S = S'$. Let us consider (2.8) term by term in the expansion of $\langle S|e^{tW}|S'\rangle$. Similarly to S and S', the gauge transformations for $S^{(k)}$ $(k = 2, \ldots, n)$ are introduced as

$$U_\sigma^{(k)} : S_i^{(k)} \longrightarrow S_i^{(k)}\sigma_i \,. \tag{3.13}$$

Then, equation (3.12) with the invariance of summation for $S^{(k)}$ leads to

$$\begin{aligned} U_\sigma U'_\sigma V_\sigma \langle S|W^n|S'\rangle &= U_\sigma U'_\sigma V_\sigma \sum_{S^{(2)}} \cdots \sum_{S^{(n)}} W(S|S^{(n)}) \prod_{k=2}^{n} W(S^{(k)}|S^{(k-1)}) \\ &= \sum_{S^{(2)}} \cdots \sum_{S^{(n)}} \left(\prod_{\ell=2}^{n} U_\sigma^{(k)} \right) U_\sigma U'_\sigma V_\sigma W(S|S^{(n)}) \prod_{k=2}^{n} W(S^{(k)}|S^{(k-1)}) \\ &= \sum_{S^{(2)}} \cdots \sum_{S^{(n)}} W(S|S^{(n)}) \prod_{k=2}^{n} W(S^{(k)}|S^{(k-1)}) \end{aligned} \tag{3.14}$$

providing the invariance (3.10).

Using equations (3.3) and (3.10) with the invariance of summation for S and S' taken into account, one can show that $\langle S_i(0)S_i(t)\rangle_K^{\mathrm{eq}}$ is gauge invariant:

$$V_\sigma \langle S_i(0)S_i(t)\rangle_K^{\mathrm{eq}} = \langle S_i(0)S_i(t)\rangle_K^{\mathrm{eq}} \,. \tag{3.15}$$

As shown in appendix A, the random average of gauge-invariant function $Q(\omega)$ can be expressed as

$$[Q(\omega)]_c = \sum_\omega \frac{D(\omega)Z(K_p, \omega)}{2^N Y(K_p)} Q(\omega) \,. \tag{3.16}$$

Thus, we obtain

$$\left[\langle S_i(0)S_i(t)\rangle_K^{\mathrm{eq}} \right]_c = \sum_\omega \frac{D(\omega)Z(K_p, \omega)}{2^N Y(K_p)} \langle S_i(0)S_i(t)\rangle_K^{\mathrm{eq}} \,. \tag{3.17}$$

On the other hand, $\langle S_i(t)\rangle_K^F$ is not gauge invariant, and is transformed as

$$\begin{aligned} V_\sigma \, \langle S_i(t)\rangle_K^F &= \sum_S U_\sigma V_\sigma \langle S|e^{tW}|F\rangle S_i \\ &= \sum_S \, \langle S|e^{tW}|\sigma\rangle S_i\sigma_i \end{aligned} \tag{3.18}$$

where we use

$$U_\sigma V_\sigma \langle S|e^{tW}|S'\rangle = U'_\sigma \langle S|e^{tW}|S'\rangle = \langle S|e^{tW}|\sigma S'\rangle \tag{3.19}$$

obtained from (3.10). The variable $\sigma S'$ represents the spin state $(\sigma_1 S'_1, \sigma_2 S'_2, \ldots, \sigma_N S'_N)$. Using equations (3.4), (3.7) and (3.18), one obtains

$$\left[\langle S_i(t)\rangle_K^F \right]_c \; = \; \sum_\omega \frac{D(\omega)Z(K_p, \omega)}{Y(K_p)} \rho_e(\sigma; K_p, \omega) \sum_S \, \langle S|e^{tW}|\sigma\rangle \; S_i\sigma_i \,. \tag{3.20}$$

Since the left-hand side of (3.20) is independent of σ, one can take the summation over all σ and divide by 2^N:

$$\left[\langle S_i(t)\rangle_K^F \right]_c \; = \; \sum_\omega \frac{D(\omega)Z(K_p, \omega)}{2^N Y(K_p)} \langle S_i(0)S_i(t)\rangle_K^{K_p} \,. \tag{3.21}$$

Then, from equation (3.16), we have an exact relation

$$\left[\langle S_i(t)\rangle_K^F \right]_c \; = \; \left[\langle S_i(0)S_i(t)\rangle_K^{K_p} \right]_c \,. \tag{3.22}$$

The function $\langle S_i(0)S_i(t)\rangle_K^{K_p}$ is the non-equilibrium autocorrelation function defined in (2.17); the temperature at $t=0$ is $J/k_\mathrm{B}K_p$. Similarly, equation (3.3) leads to

$$\left[\langle\mathcal{H}(t)\rangle_K^F\right]_\mathrm{c} = \left[\langle\mathcal{H}(t)\rangle_K^{K_p}\right]_\mathrm{c}. \tag{3.23}$$

4. Discussions

Using the rigorous relations obtained above, we discuss the exact physical properties of the Ising spin glasses.

4.1. Dynamical property in the spin-glass phase

First, let us consider the dynamics in the region with sufficiently strong randomness (small K_p) and sufficiently low temperatures (large K), where the SG phase appears if any. In such a region, $\left[\langle S_i(0)S_i(t)\rangle_K^{K_p}\right]_c$ and $\left[\langle\mathcal{H}(t)\rangle_K^{K_p}\right]_c$ describe a non-equilibrium process in the temperature $J/k_\mathrm{B}K$ relaxed from a supercooled state (cooled immediately from a high temperature with K_p to a low temperature with K); note that we assume nothing about the stability of the supercooled state., On the other hand, $\left[\langle S_i(t)\rangle_K^F\right]_c$ and $\left[\langle\mathcal{H}(t)\rangle_K^F\right]_c$ are regarded as the remanent magnetization and the exchange energy in zero field relaxed from the strong-field limit. Thus, equations (3.22) and (3.23) imply the exact equivalence of non-equilibrium relaxations from the strong-field field limit and from a supercooled state.

4.2. Dynamical property on the Nishimori line

Next, we examine the dynamics on the Nishimori line, $K=K_p$, where the non-equilibrium relaxation $\langle\cdots\rangle_K^{K_p}$ coincides with the equilibrium one $\langle\cdots\rangle_{K_p}^{\mathrm{eq}}$. From equations (3.17) and (3.22), we have

$$\left[\langle S_i(t)\rangle_{K_p}^F\right]_\mathrm{c} = \left[\langle S_i(0)S_i(t)\rangle_{K_p}^{\mathrm{eq}}\right]_\mathrm{c} \tag{4.1}$$

which indicates the equivalence of the FM and the SG orders. Since, in the SG phase, the SG order remains finite while the FM order disappears, this implies the absence of the SG phase on the Nishimori line as in the analysis of the static correlations [14].

The exchange energy of the initial state F is given by

$$\left[\langle\mathcal{H}(0)\rangle_K^F\right]_\mathrm{c} = -J\frac{\partial}{\partial K_p}\ln Y(K_p) \tag{4.2}$$

which is equal to the equilibrium energy on $K=K_p$ [14, 16]. It is easy to see that, if $K=K_p$, the right-hand side of (3.23) is independent of time t and equal to the static equilibrium average

$$\left[\langle\mathcal{H}(t)\rangle_{K_p}^F\right]_\mathrm{c} = \left[-J\sum_{\langle ij\rangle}\omega_{ij}\right]_\mathrm{c} = -J\frac{\partial}{\partial K_p}\ln Y(K_p). \tag{4.3}$$

Since the equilibrium energy is an increasing function of the temperature, we expect that $\left[\langle\mathcal{H}(t)\rangle_K^F\right]_c$ is an increasing function of t above the Nishimori line and a decreasing one below it.

The above result suggests a practical definition of the Nishimori line in real spin glasses. Actually, it is difficult to apply the present method directly to real materials because of the lack of gauge symmetry. However, for every spin-glass system, one can determine the special line defined below in the temperature–randomness phase diagram, even if it is not gauge-symmetric. Let us consider a total system composed of a spin-glass material with a non-magnetic system at temperature T left in a sufficiently strong field for a long time. The non-magnetic system is supposed to have a comparable heat capacity to the spin-glass material, and plays a role of heat bath. If the field is strong enough, the internal energy of the spin-glass material is almost independent of the temperature. When the field is suppressed from the system adiabatically, the spin-glass material absorbs heat from the heat bath if $T > J/K_B K_p$ (above the Nishimori line), and emits it if $T < J/K_B K_p$. Since this adiabatic process affects nothing directly on the non-magnetic system, the temperature of the total system decreases above the Nishimori line and increases below it. The former case shows a usual adiabatic cooling process whereas the latter one is an unusual heating process. Therefore, the Nishimori line can be defined by the border between such heating and cooling processes. It is an interesting problem in future to check the generality of the theory to real spin glasses by using the line defined above.

4.3. Absence of re-entrant transition

Finally, let us discuss the asymptotic behaviour of $[\langle S_i(0)S_i(t)\rangle_K^{K_p}]_c$ to show the verticality of the FM phase boundary in the case that the SG phase exists. In contrast with the results presented above, the following argument is plausible but non-rigorous. We use the argument introduced in the mean-field field theory [1, 2, 20, 21], however, no assumptions are necessary for the structure of the phase space. Thus, for generality, the possibility of several pure states is considered; the number of them is just one in the PM phase and two in the FM phase. We define a typical persistent time τ_N depending on (K, K_p) for large but finite systems, in which the system stays in one pure state. In the case of ultrametric space, there are several values of persistent times, and τ_N is the shortest of them. In any phases, τ_N diverges as $N \to \infty$. At time t with $\tau_N > t \gg 1$, since the system stays in the pure state $\alpha' \equiv \alpha(S')$ in which the initial state S' belongs to, the matrix $\langle S|e^{tW}|S'\rangle$ approaches the equilibrium distribution restricted in α', i.e.

$$\langle S|e^{tW}|S'\rangle \longrightarrow \rho_e(S; K)\delta_{\alpha',\alpha(S)}/w_K^{\alpha'} \tag{4.4}$$

where

$$w_K^{\alpha'} = \sum_S \rho_e(S; K)\delta_{\alpha',\alpha(S)} \tag{4.5}$$

is the weight of the pure state α' at K. Thus, from (2.17), we have

$$\langle S_i(0)S_i(\tau_N)\rangle_K^{K_p} = \sum_\alpha w_K^\alpha m_K^\alpha m_{K_p}^\alpha \tag{4.6}$$

where

$$m_K^\alpha = \sum_S S_i\,\rho_e(S; K)\delta_{\alpha,\alpha(S)}/w_K^\alpha \tag{4.7}$$

is the local magnetization at K in the pure state α. The summation of the pure states is taken over all those appearing at K in a given configuration. It is difficult to perform the random averaging of (4.6) exactly. We suppose that finite contribution in the random average comes from typical configurations in $P(\omega; K_p)$ if the system is large enough. Then, we obtain the main contribution in the random average as

$$\left[\langle S_i(0)S_i(\tau_N)\rangle_K^{K_p}\right]_c \sim \sum_\alpha \bar{w}_{K,K_p}^\alpha \bar{m}_{K,K_p}^\alpha \bar{m}_{K_p,K_p}^\alpha \tag{4.8}$$

where $\bar{w}_{K,K_p}^\alpha$ and $\bar{m}_{K,K_p}^\alpha$ are the quantities typical at (K, K_p). In the PM phase, the probability of existing non-zero local magnetization vanishes in the thermodynamic limit, whereas it remains finite in ordered phases (FM or SG). We also suppose that the direction of any local magnetization does not change with the temperature in the ordered phases, providing $\bar{m}^{\alpha}_{K,K_p}\bar{m}^{\alpha}_{K_p,K_p} \geqslant 0$. Accordingly, $\left[\langle S_i(0)S_i(\tau_N)\rangle_K^{K_p}\right]_c$ which is equal to the FM order parameter, remains finite if and only if both (K, K_p) and (K_p, K_p) are in ordered phases. This provides the verticality of the FM phase boundary implying the absence of re-entrant transition.

Although the above argument for the verticality is not rigorous, the same result can be derived by the following qualitative argument. Since the long-time limit of such correlation function is expected to behave like a kind of order parameter, (i) *the asymptotic behaviour should be prescribed by both phases where the initial point* (K_p, K_p) *and the final point* (K, K_p) *locate.* It is helpful to consider the modified model [15, 16] whose Hamiltonian is identical with the original model, whereas the bond distribution including a fixed constant a,

$$P_a(\omega; K_p) = P(\omega; K_p + a)\frac{Z(K_p, \omega)}{Z(K_p + a, \omega)}\frac{Y(K_p + a)}{Y(K_p)} \tag{4.9}$$

is different from the original model ($a = 0$ case). In appendix B, we show that average of any gauge-invariant quantity cannot distinguish the FM and the SG phases in the present systems using the exact property of the modified model. Gauge-invariant quantities can indicate only the boundary of PM phase, the onset of $q_{\mathrm{EA}} = 0$. Since $\langle S_i(0)S_i(t)\rangle_K^{K_p}$ is gauge invariant, (ii) *the asymptotic behaviour does not change even when* (K, K_p) *locate at the* FM–SG *phase boundary if* (K_p, K_p) *is not on the phase boundary.* On the other hand, it approaches the FM order parameter because of (3.22), implying that (iii) *the asymptotic behaviour changes depending on if* (K, K_p) *is* FM *or not.* At a glance, this contradicts with (ii). However, they are not in conflict with each other if the Nishimori line intersects the multicritical point and the FM phase boundary is vertical in the temperature below it; in such case, (K_p, K_p) is on the FM–PM boundary whenever (K, K_p) locates at the FM–SG boundary. This asymptotic behaviour is consistent with 4.8, and concludes the absence of re-entrant transition.

5. Remarks

We have applied the gauge transformation to dynamical systems of Ising spin glasses and derived some exact relations in dynamical quantities. The equivalence of non-equilibrium relaxations in the SG phase has been shown between a kind of field-cooled remanent magnetization and the dynamical structure factor in a supercooled state like real glasses. On the Nishimori line, we found that the equilibrium relaxation coincides with the non-equilibrium one from the strong-field field limit. We propose a practical way to define the Nishimori line in real spin glasses. The absence of re-entrant transition from the FM phase to the SG has been confirmed by the present dynamical argument. The same method can be applied to other dynamical quantities such as the AC susceptibility.

Although we restricted the dynamics to the Metropolis or the Glauber ones, the present method is applicable to the other ones with the invariance (3.11). It would be extended to other gauge-symmetric systems, which have the same transformation properties as equations (3.3) and (3.4), e.g. the gauge-glass system with the O(2) symmetry.

Appendix A. Random average of gauge-invariant quantity

To understand the main technique of the gauge transformation, we show the derivation of (3.16) following [16]. From the invariance of summation (3.7),

$$\sum_{\omega} P(\omega; K_p)Q(\omega) = \sum_{\omega} V_\sigma P(\omega; K_p)Q(\omega) \tag{A.1}$$

holds for any σ. Using equation (3.4) and the gauge invariance of $Q(\omega)$, one obtains

$$[Q(\omega)]_c = \sum_{\omega} \frac{D(\omega)Z(K_p, \omega)}{Y(K_p)} \rho_e(\sigma; K_p, \omega)Q(\omega) . \tag{A.2}$$

Since the left-hand side of (A.2) is independent of σ, the right-hand side is unchanged if one takes the summation over all σ and divide by 2^N:

$$\begin{aligned} [Q(\omega)]_c &= \frac{1}{2^N} \sum_{\sigma} \sum_{\omega} \frac{D(\omega)Z(K_p, \omega)}{Y(K_p)} \rho_e(\sigma; K_p, \omega)\, Q(\omega) \\ &= \sum_{\omega} \frac{D(\omega)Z(K_p, \omega)}{2^N Y(K_p)} Q(\omega) . \end{aligned} \tag{A.3}$$

This provides equation (3.16).

Appendix B. Modified model

In this appendix, we summarize the properties of the modified model (4.9) obtained previously [15, 16]: the random average in the modified model with a is denoted by

$$\{\cdots\}_c^a \equiv \sum_{\omega} P_a(\omega; K_p) \ldots . \tag{B.1}$$

Analysing the FM and SG order parameters defined from the static FM and SG correlation functions,

$$m_a(K, K_p)^2 \equiv \lim_{R_{ij}\to\infty} \left|\left\{\langle S_i S_j\rangle_K\right\}_c^a\right| \tag{B.2}$$

$$q_a(K, K_p)^2 \equiv \lim_{R_{ij}\to\infty} \left\{\left|\langle S_i S_j\rangle_K\right|^2\right\}_c^a \tag{B.3}$$

the following properties have been found exactly for the model with a. The boundary of the paramagnetic phase, at which the edge of $q_a = 0$ locates, is unchanged with a. The ordered phase on $K = K_p + a$ must be the FM phase. The line $K = K_p + a$ is likely to intersect the multicritical point of PM, FM and SG phases. The FM phase boundary below $K = K_p + a$ is vertical or re-entrant. Further, the verticality can be shown exactly if the ordered phase, in the models with $b \neq a$, between the line $K = K_p + b$ and the non-random case ($K_p = \infty$) is FM; this assumption is quite plausible since both boundaries exhibit only PM–FM transition. Typical phase diagram for positive and negative a is shown in figure B1. These properties are coincides with those for the original model if $a = 0$.

It is shown [16] that the average of any gauge-invariant quantity is independent of a, i.e.

$$\{Q(\omega)\}_c^a = [Q(\omega)]_c . \tag{B.4}$$

Since the FM correlation function as well as the FM order parameter (B.2) are not gauge invariant, the boundary of the FM phase changes with a. Thus, the qualitative behaviour of averaged gauge-invariant quantities at (K, K_p) should not be influenced by the fact whether (K, K_p) is FM or not. In other words, any gauge-invariant quantity can not be an order parameter for the FM phase in modified models including the original one ($a = 0$).

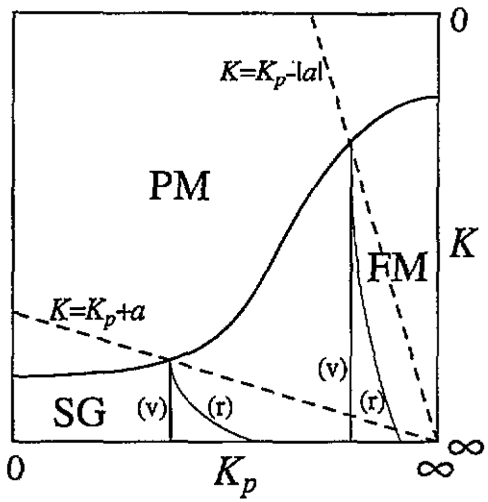

Figure B1. Phase diagram of the modified model with positive and negative a corresponding to the original model in figure 1. Possible vertical and re-entrant boundaries are indicated for both cases.

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